

BATALIN-VILKOVISKY ALGEBRA STRUCTURES ON HOCHSCHILD COHOMOLOGY.

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ABSTRACT. Let M be any compact simply-connected d -dimensional smooth manifold and let \mathbb{F} be any field. We show that the Gerstenhaber algebra structure on the Hochschild cohomology on the singular cochains of M , $HH^*(S^*(M); S^*(M))$, extends to a Batalin-Vilkovisky algebra. Such Batalin-Vilkovisky algebra was conjectured to exist and is expected to be isomorphic to the Batalin-Vilkovisky algebra on the free loop space homology on M , $H_{*+d}(LM)$ introduced by Chas and Sullivan. We also show that the negative cyclic cohomology $HC_-^*(S^*(M))$ has a Lie bracket. Such Lie bracket is expected to coincide with the Chas-Sullivan string bracket on the equivariant homology $H_*^{S^1}(LM)$.

1. INTRODUCTION

Except where specified, we work over an arbitrary field \mathbb{F} . Let M be a compact oriented d -dimensional smooth manifold. Denote by $LM := \text{map}(S^1, M)$ the free loop space on M . Chas and Sullivan [1] have shown that the shifted free loop homology $H_{*+d}(LM)$ has a structure of Batalin-Vilkovisky algebra (Definition 7). In particular, they showed that $H_{*+d}(LM)$ is a Gerstenhaber algebra (Definition 6). On the other hand, let A be a differential graded algebra. The Hochschild cohomology of A with coefficients in A , $HH^*(A; A)$, is a Gerstenhaber algebra. These two Gerstenhaber algebras are expected to be related:

Conjecture 1. *(due to [1, “dictionary” p. 5] or [3]?) If M is simply connected then there is an isomorphism of Gerstenhaber algebras $H_{*+d}(LM) \cong HH^*(S^*(M); S^*(M))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on M .*

Key words and phrases. String Topology, Batalin-Vilkovisky algebra, Gerstenhaber algebra, Hochschild cohomology, free loop space.

Félix, Thomas and Vigué-Poirrier [12, Appendix] proved that there is a linear isomorphism of lower degree d

$$(2) \quad \mathbb{D} : HH^{-p-d}(S^*(M), S^*(M)^\vee) \xrightarrow{\cong} HH^{-p}(S^*(M), S^*(M)).$$

We prove

Theorem 3. *(Theorem 21) The Connes coboundary map on $HH^*(S^*(M), S^*(M)^\vee)$ defines via the isomorphism (2) a structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra $HH^*(S^*(M), S^*(M))$.*

Assume that M is simply-connected. Jones [18] proved that there is an isomorphism

$$J : H_{p+d}(LM) \xrightarrow{\cong} HH^{-p-d}(S^*(M), S^*(M)^\vee)$$

such that the Δ operator of the Batalin-Vilkovisky algebra $H_{*+d}(LM)$ and Connes coboundary map B^\vee on $HH^{*-d}(S^*(M), S^*(M)^\vee)$ satisfies $J \circ \Delta = B^\vee \circ J$. Of course, we conjecture:

Conjecture 4. *The isomorphism*

$$\mathbb{D} \circ J : H_{p+d}(LM) \xrightarrow{\cong} HH^{-p}(S^*(M), S^*(M))$$

is a morphism of graded algebras.

Notice that Conjecture 4 implies that the composite $\mathbb{D} \circ J$ is an isomorphism of Batalin-Vilkovisky algebras between the Chas-Sullivan Batalin-Vilkovisky algebra and the Batalin-Vilkovisky algebra defined by Theorem 21. Therefore Conjecture 4 implies Conjecture 1.

Cohen and Jones [3, Theorem 3] have an isomorphism of algebras

$$H_{p+d}(LM) \xrightarrow{\cong} HH^{-p}(S^*(M), S^*(M)).$$

So one should check perhaps if the isomorphism of Cohen-Jones coincides with the isomorphism $\mathbb{D} \circ J$. Over the reals or over the rationals, two proofs of such an isomorphism of graded algebras have been given by Merkulov [24] and Félix, Thomas, Vigué-Poirrier [13].

Theorem 21 comes from a general result (Propositions 10 and 11) who shows that the Hochschild cohomology $HH^*(A; A)$ of a differential graded algebra A which is a “homotopy symmetric algebra”, is a Batalin-Vilkovisky algebra. As second application of this general result, we recover the following theorem due to Thomas Tradler.

Theorem 5. [25, Example 2.15 and Theorem 3.1] *(Corollary 18) Let A be a symmetric algebra. Then $HH^*(A, A)$ is a Batalin-Vilkovisky algebra.*

This theorem has been reproved and extended by many people [23, 19, 27, 4, 20, 21, 17, 7] (in chronological order). The last proof, the proof of Eu et Schedler [7] looks similar to ours.

Thomas Tradler gave a somehow complicated proof of the previous theorem (Corollary 18). Indeed, his goal was to prove our main theorem (Theorem 21). In [28] or in [26], Tradler and Zeinalian proved Theorem 21 but only over a field of characteristic 0 [28, “rational simplicial chain” in the abstract] or [26, Beginning of 3.1]. Costello’s result [4, Section 2.1] is also over a field of characteristic 0.

Over \mathbb{Q} , we explain how to put a Batalin-Vilkovisky algebra structure on $HH^*(S^*(M; \mathbb{Q}); S^*(M; \mathbb{Q}))$ (Corollary 19) from a slight generalisation of Corollary 18 (Theorem 17). In fact both Félix, Thomas [11] and Chen [2, Theorem 5.4] proved that the Chas-Sullivan Batalin-Vilkovisky algebra $H_{p+d}(LM; \mathbb{Q})$ is isomorphic to the Batalin-Vilkovisky algebra given by Corollary 19.

Finally, remark, that, over \mathbb{Q} , when the manifold M is formal, a consequence of Félix and Thomas work [11], is that $H_{p+d}(LM)$ is always isomorphic to the Batalin-Vilkovisky algebra $HH^*(H^*(M); H^*(M)^\vee)$ given by Corollary 18 applied to symmetric algebra $H^*(M)$. Over \mathbb{F}_2 , in [22], we showed that this is not the case. The present paper seems to explain why:

The Batalin-Vilkovisky algebra on $HH^*(S^*(M); S^*(M))$ given by Theorem 21 depends of course of the algebra $S^*(M)$ but also of a fundamental class $[m] \in HH^*(S^*(M); S^*(M)^\vee)$ which seems hard to compute. This fundamental class $[m]$ involves chain homotopies for the commutativity of the algebra $S^*(M)$.

The Batalin-Vilkovisky algebra on $HH^*(S^*(M; \mathbb{Q}); S^*(M; \mathbb{Q}))$ given by Corollary 19, depends of

- a commutative algebra, Sullivan’s cochain algebra of polynomial differential forms $A_{PL}(M)$ [9],
- and of the fundamental class $[M] \in H(A_{PL}(M))$.

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2. HOCHSCHILD HOMOLOGY AND COHOMOLOGY

Let A be a differential graded algebra. Denote by sA the suspension of A , $(sA)_i = A_{i-1}$. Let d_1 be the differential on the tensor product of complexes $A \otimes T(sA) \otimes A$. We denote the tensor product of the elements $a \in A$, $sa_1 \in sA$, \dots , $sa_k \in sA$ and $b \in A$ by $a[a_1 | \dots | a_k]b$. Let d_2 be the differential on the graded vector space $A \otimes T(sA) \otimes A$

defined by:

$$\begin{aligned} d_2 a[a_1 | \cdots | a_k] b &= (-1)^{|a|} a a_1[a_2 | \cdots | a_k] b \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[a_1 | \cdots | a_i a_{i+1} | \cdots | a_k] b \\ &\quad - (-1)^{\varepsilon_{k-1}} a[a_1 | \cdots | a_{k-1}] a_k b; \end{aligned}$$

Here $\varepsilon_i = |a| + |a_1| + \cdots + |a_i| + i$. The *bar resolution* of A , denoted $B(A; A; A)$, is the differential graded (A, A) -bimodule $(A \otimes T(sA) \otimes A, d_1 + d_2)$.

Denote by A^{op} the opposite algebra of A . Recall that any (A, A) -bimodule can be considered as a left (or right) $A \otimes A^{op}$ -module. The *Hochschild chain complex* is the complex $A \otimes_{A \otimes A^{op}} B(A; A; A)$ denoted $\mathcal{C}_*(A; A)$. Explicitly $\mathcal{C}_*(A; A)$ is the complex $(A \otimes T(sA), d_1 + d_2)$ with d_1 obtained by tensorization and

$$\begin{aligned} d_2 a[a_1 | \cdots | a_k] &= (-1)^{|a|} a a_1[a_2 | \cdots | a_k] \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[a_1 | \cdots | a_i a_{i+1} | \cdots | a_k] \\ &\quad - (-1)^{|sa_k| \varepsilon_{k-1}} a_k a[a_1 | \cdots | a_{k-1}]. \end{aligned}$$

The *Hochschild homology* is the homology of the Hochschild chain complex:

$$HH_*(A; A) := H(\mathcal{C}(A; A)).$$

Let M be a differential graded (A, A) -bimodule. The *Hochschild cochain complex* of A with coefficients in M is the complex

$$\mathcal{C}^*(A; M) = (\text{Hom}(T(sA), M), D_0 + D_1).$$

Here for $f \in \text{Hom}(T(sA), M)$, $D_0(f)([]) = d_M(f([])$, $D_1(f)([]) = 0$, and for $k \geq 1$, we have:

$$D_0(f)([a_1 | a_2 | \cdots | a_k]) = d_M(f([a_1 | a_2 | \cdots | a_k])) - \sum_{i=1}^k (-1)^{\bar{\varepsilon}_i} f([a_1 | \cdots | d_A a_i | \cdots | a_k])$$

and

$$\begin{aligned} D_1(f)([a_1 | a_2 | \cdots | a_k]) &= -(-1)^{|sa_1| |f|} a_1 f([a_2 | \cdots | a_k]) \\ &\quad - \sum_{i=2}^k (-1)^{\bar{\varepsilon}_i} f([a_1 | \cdots | a_{i-1} a_i | \cdots | a_k]) \\ &\quad + (-1)^{\bar{\varepsilon}_k} f([a_1 | a_2 | \cdots | a_{k-1}]) a_k, \end{aligned}$$

where $\bar{\varepsilon}_i = |f| + |sa_1| + |sa_2| + \cdots + |sa_{i-1}|$.

The Hochschild cohomology of A with coefficients in M is

$$HH^*(A; M) = H(\mathcal{C}^*(A; M)) = H(\text{Hom}(T(A), M), D_0 + D_1).$$

Since we work over an arbitrary field \mathbb{F} , the bar resolution $B(A; A; A) \xrightarrow{\sim} A$ is a semi-free resolution of A as an (A, A) -bimodule [9, Proposition 19.2(ii)]. Therefore the Hochschild homology of A is the differential torsion product

$$HH_*(A; A) = \text{Tor}^{A \otimes A^{op}}(A, A)$$

and the Hochschild cohomology is

$$HH^*(A; M) \cong H(\text{Hom}_{A \otimes A^{op}}(B(A; A; A), M)) = \text{Ext}_{A \otimes A^{op}}(A, M)$$

where the latter denotes the differential "Ext" in the sense of J.C. Moore (cf [8, Appendix]).

Gerstenhaber proved that the Hochschild cohomology of A with coefficients in A , $HH^*(A; A)$, is a Gerstenhaber algebra [15].

Definition 6. A *Gerstenhaber algebra* is a commutative graded algebra A equipped with a linear map $\{-, -\} : A_i \otimes A_j \rightarrow A_{i+j+1}$ of degree 1 such that:

a) the bracket $\{-, -\}$ gives A a structure of graded Lie algebra of degree 1. This means that for each a, b and $c \in A$

$$\begin{aligned} \{a, b\} &= -(-1)^{(|a|+1)(|b|+1)} \{b, a\} \text{ and} \\ \{a, \{b, c\}\} &= \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)} \{b, \{a, c\}\}. \end{aligned}$$

b) the product and the Lie bracket satisfy the following relation called the Poisson relation:

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|+1)|b|} b\{a, c\}.$$

In this paper, we show that for some algebras A , the Gerstenhaber algebra structure of $HH^*(A; A)$ extends to a Batalin-Vilkovisky algebra.

Definition 7. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra A equipped with a degree 1 linear map $\Delta : A_i \rightarrow A_{i+1}$ such that $\Delta \circ \Delta = 0$ and

$$(8) \quad \{a, b\} = (-1)^{|a|} (\Delta(a \cup b) - (\Delta a) \cup b - (-1)^{|a|} a \cup (\Delta b))$$

for a and $b \in A$.

3. THE ISOMORPHISM BETWEEN $HH^*(A; A)$ AND $HH^*(A; A^\vee)$

In this section, we first present a method that gives an isomorphism between the Hochschild cohomology of A with coefficients in A , $HH^*(A; A)$ and the Hochschild cohomology of A with coefficients in the dual A^\vee , $HH^*(A; A^\vee)$. This method is a generalisation of the

method used by Félix, Thomas and Vigué-Poirrier to obtain the isomorphism (2). Then we show that this isomorphism looks like a Poincaré duality isomorphism: this isomorphism is given by the action of the algebra $HH^*(A; A)$ on a fundamental class $[m] \in HH^*(A; A^\vee)$.

Let us first recall the definition of the action of $HH^*(A; A)$ on $HH^*(A; A^\vee)$. Let A be a (differential graded) algebra. Let M and N two A -bimodules. Let $f \in \mathcal{C}^*(A, M)$ and $g \in \mathcal{C}^*(A, N)$. We denote by $\otimes_A(f, g) \in \mathcal{C}^*(A, M \otimes_A N)$ the linear map defined by

$$\otimes_A(f, g)([a_1 | \dots | a_n]) = \sum_{p=0}^n f([a_1 | \dots | a_p]) \otimes_A g([a_{p+1} | \dots | a_n]).$$

This define a natural morphism of complexes

$$\otimes_A : \mathcal{C}^*(A, M) \otimes \mathcal{C}^*(A, N) \rightarrow \mathcal{C}^*(A, M \otimes_A N)$$

Therefore, in homology, we have a natural morphism

$$H_*(\otimes_A) : HH^*(A, M) \otimes HH^*(A, N) \rightarrow HH^*(A, M \otimes_A N)$$

If we let take $A = M$, and use the isomorphism of A -bimodules

$$A \otimes_A N \xrightarrow{\cong} N, a \otimes_A n \mapsto a.n,$$

the composite

$$(9) \quad \mathcal{C}^*(A, A) \otimes \mathcal{C}^*(A, N) \xrightarrow{\otimes_A} \mathcal{C}^*(A, A \otimes_A N) \cong \mathcal{C}^*(A, N)$$

is a left action of $\mathcal{C}^*(A, A)$ on $\mathcal{C}^*(A, N)$. In the particular case, $A = M = N$, this composite is the usual cup product on $\mathcal{C}^*(A, A)$ denoted \cup .

Denote by A^\vee the dual of A . Let $\eta : \mathbb{F} \rightarrow A$ be the unit of the algebra. Then we have a natural map

$$HH^*(\eta, A^\vee) : HH^*(A, A^\vee) \rightarrow HH^*(\mathbb{F}, A^\vee) \cong H(A^\vee).$$

Proposition 10. *Let $[m] \in HH^{-d}(A, A^\vee)$ be an element of lower degree d such that the morphism of left $H(A)$ -modules*

$$H(A) \xrightarrow{\cong} H(A^\vee), a \mapsto a.HH^{-d}(\eta, A^\vee)([m])$$

is an isomorphism. Then the action of $HH^(A, A)$ on $[m] \in HH^{-d}(A, A^\vee)$ gives the isomorphism of lower degree d of $HH^*(A, A)$ -modules*

$$HH^p(A, A) \xrightarrow{\cong} HH^{p-d}(A, A^\vee), a \mapsto a \cdot [m].$$

Proof. Let $\varepsilon_A : P \xrightarrow{\cong} A$ be a resolution of A as left $A \otimes A^{op}$ -semifree module. Let $s_A : A \xrightarrow{\cong} P$ be a morphism of left A -modules which is a section of ε_A . The morphism $HH^*(\eta, A^\vee) : HH^*(A, A^\vee) \rightarrow HH^*(\mathbb{F}, A^\vee)$ is equal to the following composite of

$$HH^*(A, A^\vee) := Ext_{A \otimes A^{op}}(A, A^\vee) \xrightarrow{Ext_{i_1}(A, A^\vee)} Ext_A(A, A^\vee)$$

and

$$Ext_A(A, A^\vee) \xrightarrow[\cong]{Ext_{\eta}(\eta, A^\vee)} Ext_{\mathbb{F}}(\mathbb{F}, A^\vee) =: HH^*(\mathbb{F}, A^\vee)$$

where $i_1 : A \hookrightarrow A \otimes A^{op}$ is the inclusion of the first factor.

Therefore, $HH^*(\eta, A^\vee)$ is the map induced in homology by the composite

$$Hom_{A \otimes A^{op}}(P, A^\vee) \xrightarrow{Hom(s_A, A^\vee)} Hom_A(A, A^\vee) \xrightarrow[\cong]{ev(1_A)} A^\vee.$$

where $ev(1_A)$ is the evaluation at the unit $1_A \in A$. This composite maps the cycle $m \in Hom_{A \otimes A^{op}}(P, A^\vee)$ to $m \circ s_A$ and then to $(m \circ s_A)(1_A)$. Since $m \circ s_A : A \rightarrow A^\vee$ maps $a \in A$ to $a \cdot ((m \circ s_A)(1))$, by hypothesis, $m \circ s_A$ is a quasi-isomorphism. Since s_A is a quasi-isomorphism, $m : P \xrightarrow{\cong} A^\vee$ is also a quasi-isomorphism.

By applying the functor $Hom_{A \otimes A^{op}}(P, -)$ to the two quasi-isomorphisms of A -bimodules

$$A \xleftarrow[\cong]{\varepsilon_A} P \xrightarrow[\cong]{m} A^\vee,$$

we obtain the quasi-isomorphism of complexes

$$Hom_{A \otimes A^{op}}(P, A) \xleftarrow[\cong]{\varepsilon_A} Hom_{A \otimes A^{op}}(P, P) \xrightarrow[\cong]{Hom_{A \otimes A^{op}}(P, m)} Hom_{A \otimes A^{op}}(P, A^\vee).$$

By applying homology, we get the desired isomorphism, since the action of $HH^*(A, A)$ on $HH^*(A, A^\vee)$ is induced by the composition map

$$Hom_{A \otimes A^{op}}(P, A^\vee) \otimes Hom_{A \otimes A^{op}}(P, P) \rightarrow Hom_{A \otimes A^{op}}(P, A^\vee)$$

$$m \otimes f \mapsto m \circ f = Hom_{A \otimes A^{op}}(P, m)(f)$$

Alternatively, the two isomorphisms

$$HH^*(A, A) \xleftarrow[\cong]{HH^*(A, \varepsilon_A)} HH^*(A, P) \xrightarrow[\cong]{HH^*(A, m)} HH^*(A, A^\vee)$$

maps ε_A (which is the unit of $HH^*(A, A)$) to $id_P : P \rightarrow P$ and then to m . They are morphisms of $HH^*(A, A)$ -modules since

$$H_*(\otimes_A) : HH^*(A, A) \otimes HH^*(A, N) \rightarrow HH^*(A, A \otimes_A N)$$

is natural with respect to N . □

4. BATALIN-VILKOVISKY ALGEBRA STRUCTURES ON HOCHSCHILD COHOMOLOGY

In this section, we explain when an isomorphism $HH^*(A; A) \cong HH^*(A; A^\vee)$ gives a Batalin-Vilkovisky algebra structure on the Gerstenhaber algebra $HH^*(A; A)$. Our proof relies on the proof of a similar result due to Ginzburg [16, Theorem 3.4.3 (ii)]. Ginzburg basically explains when an isomorphism $HH^*(A; A) \cong HH_*(A; A)$ gives a Batalin-Vilkovisky algebra structure on $HH^*(A; A)$.

Denote by B Connes boundary in the Hochschild complex $\mathcal{C}_*(A; A)$ and by B^\vee its dual in $\mathcal{C}^*(A, A^\vee) \cong \mathcal{C}_*(A; A)^\vee$. We prove:

Proposition 11. *Let $[m] \in HH^{-d}(A, A^\vee)$ such that the morphism of $HH^*(A, A)$ -modules*

$$HH^p(A, A) \xrightarrow{\cong} HH^{p-d}(A, A^\vee), \quad a \mapsto a.[m]$$

is an isomorphism. If $H_(B^\vee)([m]) = 0$ then the Gerstenhaber algebra $HH^*(A, A)$ equipped with $H_*(B^\vee)$ is a Batalin-Vilkovisky algebra.*

As we will see Proposition 11 is almost the dual of the following Proposition due to Victor Ginzburg. Recall first that the Hochschild cohomology of a (differential graded) algebra, acts on its Hochschild homology

$$HH^p(A; A) \otimes HH_d(A; A) \rightarrow HH_{p-d}(A; A) \\ \eta \otimes c \mapsto i_\eta(c) = \eta.c$$

In non-commutative geometry, the action of $\eta \in HH^*(A; A)$ on $c \in HH_*(A; A)$ is denoted by $i_\eta(c)$.

Proposition 12. [16, Theorem 3.4.3 (ii)] *Let $c \in HH_d(A, A)$ such that the morphism of $HH^*(A, A)$ -modules*

$$HH^p(A, A) \xrightarrow{\cong} HH_{d-p}(A, A), \quad \eta \mapsto \eta.c$$

is an isomorphism. If $H_(B)(c) = 0$ then the Gerstenhaber algebra $HH^*(A, A)$ equipped with $H_*(B)$ is a Batalin-Vilkovisky algebra.*

Remark 13. The condition $H_*(B)(c) = 0$ does not appear in [16, Theorem 3.4.3 (ii)] since according to Ginzburg, this condition is automatically satisfied for a Calabi-Yau algebra of dimension d . In both Propositions 11 and 12, if the condition $H_*(B^\vee)([m]) = 0$ or $H_*(B)(c) = 0$ is not satisfied, $\Delta(1)$ can be non zero and the relation (8) is replaced by the more general relation

$$\{\xi, \eta\} = (-1)^{|\xi|} [\Delta(\xi \cup \eta) - \\ (-1)^{|\xi|} \xi \cup (\Delta\eta) - (\Delta\xi) \cup \eta + (-1)^{|\xi|+|\eta|} \xi \cup \eta \cup (\Delta 1)].$$

Proof of Proposition 12. By definition the Δ operator on $HH^*(A; A)$ is given by $(\Delta a).c := B(a.c)$ for any $a \in HH^*(A; A)$. Therefore the proposition follows from the following Lemma due to Victor Ginzburg. \square

Lemma 14. [16, formula (9.3.2)] *Let A be a differential graded algebra. For any $\eta, \xi \in HH^*(A; A)$ and $c \in HH_*(A; A)$,*

$$\begin{aligned} \{\xi, \eta\}.c &= (-1)^{|\xi|} B[(\xi \cup \eta).c] - \xi.B(\eta.c) \\ &\quad + (-1)^{(|\eta|+1)(|\xi|+1)} \eta.B(\xi.c) + (-1)^{|\eta|} (\xi \cup \eta).B(c). \end{aligned}$$

Proof. Let us recall the proof of Victor Ginzburg. Denote by

$$\begin{aligned} HH^p(A; A) \otimes HH_j(A; A) &\rightarrow HH_{j-p+1}(A; A) \\ (\eta, a) &\mapsto L_\eta(a) \end{aligned}$$

the action of the suspended graded Lie algebra $HH^*(A; A)$ on $HH_*(A; A)$. Gelfand, Daletski and Tsygan [14] proved that the Gerstenhaber algebra $HH^*(A; A)$ and Connes boundary map B on $HH_*(A; A)$ form a calculus [5, p. 93]. Therefore, we have the following equalities

$$\begin{aligned} i_{\{\xi, \eta\}} &= \{L_\xi, i_\eta\} = L_\xi \circ i_\eta - (-1)^{(|\xi|+1)|\eta|} i_\eta \circ L_\xi \\ &= (-1)^{|\xi|} \{B, i_\xi\} \circ i_\eta - (-1)^{(|\xi|+1)|\eta|} i_\eta \circ (-1)^{|\xi|} \{B, i_\xi\} \\ &= (-1)^{|\xi|} B \circ i_\xi \circ i_\eta - i_\xi \circ B \circ i_\eta + (-1)^{(|\eta|+1)(|\xi|+1)} i_\eta \circ B \circ i_\xi + (-1)^{|\eta|(|\xi|+1)} i_\eta \circ i_\xi \circ B \\ &= (-1)^{|\xi|} B \circ i_{\xi \cup \eta} - i_\xi \circ B \circ i_\eta + (-1)^{(|\eta|+1)(|\xi|+1)} i_\eta \circ B \circ i_\xi + (-1)^{|\eta|} i_{\xi \cup \eta} \circ B. \end{aligned}$$

By applying this equality of operators to c , we obtain the Lemma. \square

We now prove the following Lemma which is the dual of Lemma 14.

Lemma 15. *Let A be a differential graded algebra. For any $\eta, \xi \in HH^*(A; A)$ and $m \in HH^*(A; A^\vee)$,*

$$\begin{aligned} \{\xi, \eta\}.m &= (-1)^{|\xi|} B^\vee[(\xi \cup \eta).m] - \xi.B^\vee(\eta.m) \\ &\quad + (-1)^{(|\eta|+1)(|\xi|+1)} \eta.B^\vee(\xi.m) + (-1)^{|\eta|} (\xi \cup \eta).B^\vee(m). \end{aligned}$$

Proof. The action of $HH^*(A; A)$ on $HH_*(A; A)$ comes from a (right) action of the $\mathcal{C}^*(A; A)$ on $\mathcal{C}_*(A; A)$ given by

$$\mathcal{C}_*(A; A) \otimes \mathcal{C}^*(A; A) \rightarrow \mathcal{C}_*(A; A)$$

$$(m[a_1 | \dots | a_n], f) \mapsto i_f(m[a_1 | \dots | a_n]) := \sum_{p=0}^n (m.f[a_1 | \dots | a_p])[a_{p+1} | \dots | a_n].$$

Therefore $\mathcal{C}^*(A; A)$ acts on the left on the dual $\mathcal{C}_*(A; A)^\vee$. Explicitly, the action is given by

$$\mathcal{C}^*(A; A) \otimes \mathcal{C}_*(A; A)^\vee \rightarrow \mathcal{C}_*(A; A)^\vee$$

$$(f, \varphi) \mapsto \varphi \circ i_f.$$

Through the canonical isomorphism $\mathcal{C}(A; A^\vee) \xrightarrow{\cong} \mathcal{C}_*(A; A)^\vee$, $g \mapsto \varphi$ defined by $\varphi(m[a_1 | \dots | a_n]) := (g[a_1 | \dots | a_n])(m)$, this left action coincides with the left action defined by the composite (9).

Let us precise our sign convention: we define B^\vee by $B^\vee(m) := (-1)^{|m|} m \circ B$. Denote by ε the sign $(-1)^{|m|(|\xi|+|\eta|+1)}$. For any $m \in HH_*(A; A)^\vee$, we have the following equalities:

$$\begin{aligned} m(\{\xi, \eta\}.c) &= \varepsilon(\{\xi, \eta\}.m)(c), \\ (-1)^{|\xi|} m \circ B[(\xi \cup \eta).c] &= (-1)^{|\xi|+|m|} [B^\vee(m)][(\xi \cup \eta).c] = \varepsilon(-1)^{|\eta|} [(\xi \cup \eta).B^\vee(m)](c), \\ -m[\xi.B(\eta.c)] &= (-1)^{1+|m||\xi|} [\xi.m] \circ B(\eta.c) = \\ (-1)^{1+|m||\xi|+|\xi|+|m|} [B^\vee(\xi.m)](\eta.c) &= \varepsilon(-1)^{(|\eta|+1)(|\xi|+1)} [\eta.B^\vee(\xi.m)](c), \end{aligned}$$

by exchanging ξ and η ,

$$(-1)^{(|\eta|+1)(|\xi|+1)} m[\eta.B(\xi.c)] = -\varepsilon[\xi.B^\vee(\eta.m)](c),$$

$$\begin{aligned} (-1)^{|\eta|} m[(\xi \cup \eta).B(c)] &= \varepsilon(-1)^{|\eta|+|m|} [(\xi \cup \eta).m] \circ B(c) = \\ &= \varepsilon(-1)^{|\xi|} B^\vee[(\xi \cup \eta).m](c). \end{aligned}$$

Therefore by evaluating the linear form m on the terms of the equation given by Lemma 14, we obtain the desired equality. \square

Remark 16. The equality in Lemma 15 is the same as the equality in Lemma 14. In fact, alternatively, to prove Lemma 15, we could have proved that the Gerstenhaber algebra $HH^*(A; A)$ and the dual of Connes boundary map B^\vee on $HH^*(A; A^\vee)$ form a calculus. Indeed, in the proof of Lemma 14, we have remarked that the desired equality holds for any calculus.

Proof of Proposition 11. By definition the Δ operator on $HH^*(A; A)$ is given by $(\Delta a).m := B^\vee(a.m)$ for any $a \in HH^*(A; A)$. Therefore the proposition follows from Lemma 15. \square

5. APPLICATIONS

As first application of Proposition 11, we show

Theorem 17. *Let A be an algebra equipped with a degree d quasi-isomorphism of A -bimodules $\Theta : A \xrightarrow{\cong} A^\vee$ between A and its dual $\text{Hom}(A, \mathbb{F})$. Then the Connes coboundary map on $HH^*(A, A^\vee)$ defines via the isomorphism $HH^*(A, \Theta) : HH^p(A, A) \xrightarrow{\cong} HH^{p-d}(A, A^\vee)$ a structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra $HH^*(A, A)$.*

In representation theory [6], an (ungraded) algebra A is *symmetric* if A is equipped with an isomorphism of A -bimodules $\Theta : A \xrightarrow{\cong} A^\vee$ between A and its dual $\text{Hom}(A, \mathbb{F})$. The following Corollary is implicit in [25] and was for the first time explicited in [23, Theorem 1.6].

Corollary 18. [25, 23] *Let A be a symmetric algebra. Then $HH^*(A, A)$ is a Batalin-Vilkovisky algebra.*

In [19] or [27, Corollary 3.4] or [4, Section 1.4] or [20, Theorem B] or [21, Section 11.6] or [17], this Batalin-Vilkovisky algebra structure on $HH^*(A, A)$ extends to a structure of algebra on the Hochschild cochain complex $\mathcal{C}^*(A, A)$ over various operads or PROPs: the so-called cyclic Deligne conjecture.

Proof of Theorem 17. Let $\varepsilon_A : P := B(A; A; A) \xrightarrow{\cong} A$ be the bar resolution of A . Denote by m the composite $P \xrightarrow{\varepsilon_A} A \xrightarrow{\Theta} A^\vee$. Since m commutes with the differential, m is a cycle in $\text{Hom}_{A \otimes A^{op}}(P, A)$. As we saw in the proof of Proposition 10, the composite $HH^*(A, m) \circ HH^*(A, \varepsilon_A)^{-1} :$

$$HH^*(A, A) \xrightarrow[\cong]{HH^*(A, \varepsilon_A)} HH^*(A, P) \xrightarrow[\cong]{HH^*(A, m)} HH^*(A, A^\vee)$$

coincides with the morphism of left $HH^*(A; A)$ -modules

$$HH^p(A; A) \xrightarrow{\cong} HH^{p-d}(A; A^\vee), a \mapsto a \cdot m.$$

By definition of m , this composite is also $HH^*(A, \Theta)$.

Denote by $\varepsilon_{BA} : TsA \rightarrow \mathbb{F}$ the canonical projection whose kernel is T^+sA . Since $\varepsilon_A : B(A; A; A) \rightarrow A$ is the composite of $A \otimes \varepsilon_{BA} \otimes A$ and of the multiplication on A

$$A \otimes TsA \otimes A \rightarrow A \otimes \mathbb{F} \otimes A \cong A \otimes A \rightarrow A,$$

the canonical isomorphisms of complexes

$$\text{Hom}_{A \otimes A^{op}}(B(A; A; A), A^\vee) \cong \mathcal{C}^*(A; A^\vee) \cong \mathcal{C}_*(A; A)^\vee$$

map m to the linear form on $\mathcal{C}_*(A; A)$:

$$\Theta(1) \otimes \varepsilon_{BA} : A \otimes TsA \rightarrow \mathbb{F} \otimes \mathbb{F} \cong \mathbb{F}.$$

Connes (normalized or not) boundary map $B : \mathcal{C}_*(A; A) \rightarrow \mathcal{C}_*(A; A)$ factorizes through $A \otimes T^+sA$. So $B^\vee(\Theta(1) \otimes \varepsilon_{BA}) = \pm(\Theta(1) \otimes \varepsilon_{BA}) \circ B = 0$. Therefore, we can apply Proposition 11

Remark: In the case of Corollary 18, m correspond to a trace $\Theta(1) \in \mathcal{C}^0(A; A^\vee)$. Since $H(B^\vee) : HH^p(A; A^\vee) \rightarrow HH^{p-1}(A; A^\vee)$ decreases (upper) degrees and $HH^p(A; A^\vee) = 0$ for $p < 0$, it is obvious that $H(B^\vee)(\Theta(1)) = 0$. \square

Working, with rational coefficients, we easily obtain

Corollary 19. [28] *The Hochschild cohomology*

$$HH^*(S^*(M; \mathbb{Q}); S^*(M; \mathbb{Q})) \cong HH^{*-d}(S^*(M; \mathbb{Q}); S^*(M; \mathbb{Q}))$$

is a Batalin-Vilkovisky algebra.

Tradler and Zeinalian [28] give a proof of this result. Here is a shorter proof, although we don't claim that we have obtained the same Batalin-Vilkovisky algebra. It should not be difficult to see that the Batalin-Vilkovisky algebra given by Corollary 19 coincides with the Batalin-Vilkovisky algebra given by our main theorem (Theorem 21) in the case of the field \mathbb{Q} . Therefore, one could deduce Corollary 19 from Theorem 21. But it is much more simple to give a separate proof of Corollary 19. As we would like to emphasize in this paper, the rational case is much more simple than the case of a field \mathbb{F} of characteristic p different from 0.

Proof of Corollary 19. Since we are working over \mathbb{Q} , there exists quasi-isomorphisms of algebras [9, Corollary 10.10] $S^*(M; \mathbb{Q}) \xrightarrow{\cong} D(M) \xleftarrow{\cong} A_{PL}(M)$ where $A_{PL}(M)$ is a commutative (differential graded) algebra. Since the Gerstenhaber algebra structure on Hochschild cohomology is preserved by quasi-isomorphism of algebras [10, Theorem 3], we obtain an isomorphism of Gerstenhaber algebras

$$HH^*(S^*(M; \mathbb{Q}); S^*(M; \mathbb{Q})) \cong HH^*(A_{PL}(M); A_{PL}(M)).$$

Since $H(A_{PL}(M)) \cong H^*(M; \mathbb{Q})$, Poincaré duality induces a quasi-isomorphism of $A_{PL}(M)$ -modules, and so of $A_{PL}(M)$ -bimodules, since the algebra $A_{PL}(M)$ is commutative:

$$A_{PL}(M) \xrightarrow[\cong]{\cap [M]} A_{PL}(M)^\vee.$$

By applying Theorem 17, we obtain that

$$HH^*(A_{PL}(M); A_{PL}(M)) \cong HH^{*-d}(A_{PL}(M); A_{PL}(M)^\vee)$$

is a Batalin-Vilkovisky algebra. □

In [11] and [2, Theorem 5.4], it is shown that the Batalin-Vilkovisky algebra $H_{p+d}(LM; \mathbb{Q})$ of Chas and Sullivan is isomorphic to the Batalin-Vilkovisky algebra on

$$HH^{-p}(S^*(M; \mathbb{Q}); S^*(M; \mathbb{Q})) \cong HH^{-p-d}(A_{PL}(M); A_{PL}(M)^\vee).$$

given by Corollary 19.

Recall the following theorem due to Félix, Thomas and Vigué-Poirrier.

Theorem 20. [12, Appendix] *Let M be a compact connected oriented d -dimensional smooth manifold. Then there is an isomorphism of lower degree d*

$$\mathbb{D}^{-1} : HH^p(S^*(M), S^*(M)) \xrightarrow{\cong} HH^{p-d}(S^*(M), S^*(M)^\vee).$$

As second application of Propositions 10 and 11, we will recover the isomorphism of Félix, Thomas and Vigué-Poirrier and prove our main theorem:

Theorem 21. *Let M be a compact connected oriented d -dimensional smooth manifold. Let $[M] \in H_d(M)$ be its fundamental class. Then*

1) *For any $a \in HH^*(S^*(M), S^*(M))$, the image of a by \mathbb{D}^{-1} is given by the action of a on $(J \circ H_*(s))([M])$:*

$$\mathbb{D}^{-1}(a) = a \cdot (J \circ H_*(s))([M]).$$

2) *The Gerstenhaber algebra structure on $HH^*(S^*(M); S^*(M))$ and Connes coboundary map $H(B^\vee)$ on $HH^*(S^*(M); S^*(M)^\vee)$ defines via the isomorphism \mathbb{D}^{-1} a structure of Batalin-Vilkovisky algebra.*

Here s denotes $s : M \hookrightarrow LM$ the inclusion of the constant loops into LM . Recall that $J : H_*(LM) \rightarrow HH^*(S^*(M), S^*(M)^\vee)$ is the morphism introduced by Jones in [18]. If M is supposed to be simply connected, then J is an isomorphism.

Proof of Theorem 20 and of Theorem 21. We first follow basically [12, Appendix]. Denote by $ev : LM \rightrightarrows M$, $l \mapsto l(0)$ the evaluation map. The morphism J of Jones fits into the commutative triangle.

$$\begin{array}{ccc} H_*(LM) & \xrightarrow{J} & HH^*(S^*(M), S^*(M)^\vee) \\ & \searrow H_*(ev) & \swarrow HH^*(\eta, S^*(M)^\vee) \\ & H_*(M) & \end{array}$$

Since s is a section of the evaluation map ev , $J \circ H_*(s)$ is a section of $HH^*(\eta, S^*(M)^\vee)$. Therefore $HH^*(\eta, S^*(M)^\vee) \circ J \circ H_*(s)([M]) = [M]$.

By Poincaré duality, the composite of the two morphisms of $H^*(M)$ -module

$$H^*(M) \xrightarrow{\cap [M]} H_*(M) \cong H(S^*(M)^\vee), a \mapsto a \cap [M] \mapsto a \cdot [M].$$

is an isomorphism of lower degree d . Therefore by applying Proposition 10 to $[m] := J \circ H_*(s)([M])$, we obtain Theorem 20 and part 1) of Theorem 21.

Consider M equipped with the trivial S^1 -action. The section $s : M \hookrightarrow LM$ is S^1 -equivariant. Therefore $\Delta(H_*(s)([M])) = 0$. Recall

that the Jones morphism J satisfies $J \circ \Delta = H_*(B^\vee) \circ J$. Therefore, since $(H_*(B^\vee) \circ J \circ H_*(s))([M]) = 0$, by applying Proposition 11, we obtain part 2) of Theorem 21. \square

Remark 22. Part 1) of Theorem 21 means exactly that the morphism

$$\mathbb{D}^{-1} : HH^p(S^*(M), S^*(M)) \xrightarrow{\cong} HH^{p-d}(S^*(M), S^*(M)^\vee)$$

is the unique morphism of $HH^*(S^*(M), S^*(M))$ -modules such that the composite

$$J^{-1} \circ \mathbb{D}^{-1} : HH^{-p}(S^*(M), S^*(M)) \xrightarrow{\cong} H_{p+d}(LM)$$

respects the units of the algebras. We conjecture (Conjecture 4) that $J^{-1} \circ \mathbb{D}^{-1}$ respects also the products.

6. CYCLIC HOMOLOGY

In this section, we prove

Corollary 23. *Let M be a compact oriented smooth d -dimensional manifold. Then the negative cyclic cohomology on the singular cochains of M , $HC_-^*(S^*(M))$, is a graded Lie algebra of lower degree $2 - d$.*

If M is simply-connected, Jones [18] proved that there is an isomorphism

$$H_*^{S^1}(LM) \xrightarrow{\cong} HC_-^*(S^*(M)).$$

In [1], Chas and Sullivan defined a Lie bracket, called the string bracket

$$\{ , \} : H_p^{S^1}(LM) \otimes H_q^{S^1}(LM) \rightarrow H_{p+q+2-d}^{S^1}(LM)$$

Of course, we expect the two a priori different brackets to be related:

Conjecture 24. *The Jones isomorphism*

$$H_*^{S^1}(LM) \xrightarrow{\cong} HC_-^*(S^*(M))$$

is an isomorphism of graded Lie algebras between Chas-Sullivan string bracket and the Lie bracket defined in Corollary 23.

Corollary 23 follows directly from Theorem 21 and from the following proposition. In [23, Corollary 1.7 and Section 7], we proved that if A is a symmetric algebra then its negative cyclic cohomology $HC_-^*(A)$ is a graded Lie algebra of lower degree 2. In fact, we proved more generally

Proposition 25. *If the Hochschild cohomology of a (differential graded) algebra A , $HH^*(A; A^\vee)$, equipped with $H_*(B^\vee)$, has a Batalin-Vilkovisky algebra structure of degree $-d$ then its negative cyclic cohomology $HC_-^*(A)$ is a graded Lie algebra of lower degree $2-d$.*

Proof. Apply [23, Proposition 7.1] to the mixed complex $\mathcal{C}^*(A; A^\vee)$ (desuspended d -times in order to take into account the degree d shift). By definition, $HC_-^*(A)$ is the differential torsion product

$$\mathrm{Tor}^{H^*(S^1)}(\mathcal{C}^*(A; A^\vee), \mathbb{F}). \quad \square$$

Another interesting particular case of [23, Proposition 7.1] is the following proposition.

Proposition 26. *If the Hochschild homology of an algebra A , $HH_*(A; A)$, equipped with Connes boundary map B , has a Batalin-Vilkovisky algebra structure then its cyclic homology $HC_*(A)$ is a graded Lie algebra of lower degree 2.*

Proof. Apply [23, Proposition 7.1] to the mixed complex $\mathcal{C}_*(A; A)$. By definition, $HC_*(A)$ is the differential torsion product

$$\mathrm{Tor}^{H^*(S^1)}(\mathcal{C}_*(A; A), \mathbb{F}). \quad \square$$

Remark that in fact, these graded Lie algebra structures extend to Lie_∞ -algebra structures like the Chas-Sullivan string bracket [1, Theorem 6.2 and Corollary 6.3].

Chas-Sullivan string bracket is defined using Gysin long exact sequence. The bracket given by Corollary 23 is defined similarly using Connes long exact sequence. Jones [18] proved that Gysin and Connes long exact sequences are isomorphic. Therefore Conjecture 4 implies Conjecture 24, since as we explained in the introduction, Conjecture 4 implies that the Jones isomorphism

$$J : H_{p+d}(LM) \xrightarrow{\cong} HH^{-p-d}(S^*(M), S^*(M)^\vee)$$

is an isomorphism of Batalin-Vilkovisky algebras.

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